$\begin{bmatrix} \uparrow\ b_1 \rightarrow & \uparrow \\ a_1 \cdots & a_n \\ \downarrow \leftarrow b_m \rightarrow & \downarrow \end{bmatrix}$ $\text{Col}A = \text{Span}\{a_1 \cdots a_n\} \to$ in $\mathbb{R}^n$

$\text{row}A = \text{Span}\{b_1 \cdots b_m\}$ in $\mathbb{R}^m$

row column $m \times n$ — can be in $\text{Nul}A \in \mathbb{R}^n$

$\underline{A}\ \underline{x} = \underline{b} \to$ can be in $\text{Col}A \in \mathbb{R}^m$

$$\begin{bmatrix} \leftarrow n \rightarrow \\ m \quad \underline{A} \\ \downarrow \end{bmatrix}\begin{bmatrix} n \end{bmatrix} = \begin{bmatrix} m \end{bmatrix}$$

$T: \mathbb{R}^{n \to \text{domain}} \to \mathbb{R}^{m \to \text{codomain}}$ $\quad T(e_i) = a_i$ (i column of transformation matrix)

$A^T\underline{b} = A^TA\hat{\underline{x}}$ } set of least squares sol'ns = set of sol'ns to... always... if A invertible then only 1 element in this set.

A^TA always square matrix...

$A^T\underline{x} = 0 \Rightarrow (A^T\underline{x})^T = \underline{0}^T \Rightarrow \underline{x}^TA = \underline{0}^T$ $\quad \text{col}(A^T) = \text{row}A$, $\text{rank}A = \text{rank}A^T$

Adv

one to one	onto	both
$m \geq n$	$m \leq n$	$n = m$
pivot in each column (n)	pivot in each row (m)	n pivot columns / rows
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
Columns of A are linearly independent. $A\underline{x} = \underline{0}$ has only trivial sol'n. $T(\underline{x}) = \underline{b}$ has at most one sol'n for each $\underline{b} \in \mathbb{R}^m$. $-\text{Nul}A = \{0\}$ $-\dim\text{Nul}A = 0$ $-\text{rank}A = n$	Columns of $\underline{A}$ span $\mathbb{R}^m$. $A\underline{x} = \underline{b}$ has at least one sol'n for each $b \in \mathbb{R}^m$. Range of T = codomain of T. $-\text{Col}A = \mathbb{R}^m$, $\dim\text{Col}A = m$. $\text{rank}A = m$	A is invertible, $A \sim I_n$. $A\underline{x} = \underline{b}$ has a unique sol'n for each $\underline{b} \in \mathbb{R}^n$. A^T is invertible, $\lambda \neq 0$. Columns of A form basis for $\mathbb{R}^n$. $\text{Col}A = \mathbb{R}^n$. $\dim\text{Col}A = n$, $\text{rank}A = n$. $\text{Null}A = \{0\}$ $\dim\text{Null}A = 0$. linear transformation is invertible so A^{-1} exists $[T^{-1}(T(x)) = x]$

How to find determinant

$\to \det A = \sum_{k=1}^{n} a_{ik}(-1)^{i+k}\det A_{ik}$, $= C_{ik}$ (remove ith row and jth column.

↳ use cofactor expansion w/ row if ith row has 0s.

$\to \det A = \sum_{k=1}^{n} a_{kj}(-1)^{k+j}\det A_{kj} \to$ use if kth row has 0s.

$\to$ if upper triangular $\det A = a_{11}a_{22}\cdots a_{nn}$ $\quad A = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$

$\det A^{-1} = 1/\det A$ $\quad$ # row interchanges

$\Leftrightarrow \det A \neq 0 = (-1)$ $= \det BA$. (product of pivots in echelon form)

$\det A = \det A^T$; $\det AB = (\det A)(\det B)$; $\det(A+B) \neq \det A + \det B$

$\to$ Can row reduce A to B and find $\det B$, $\det A$ changes

↳ row interchange makes: $\det B = -\det A$

↳ Multiply by scalar k: $\det B = k\det A$ $\quad \begin{vmatrix} kk \\ 02 \end{vmatrix} = k\begin{vmatrix} 11 \\ 02 \end{vmatrix} = k \cdot 1 \cdot 2$

↳ multiple of the row added to another $\det A = \det B$

$\text{Det}\,EA = (\text{Det}\,E)(\text{Det}\,A)$, $\text{Det}\,E = -1, 1,$ or k depending on row operation.

$\Leftrightarrow$ Cramer's rule: unique sol'n $x_i = \dfrac{\det(A_i(\underline{b}))}{\det A} \to b$ put in ith column of A.

$\det kA = k^n\det A$ ($A = n \times n$)

Subspace of $\mathbb{R}^n$, any set H that satisfies $-\underline{0}$ is in H

↳ zero subspace $H = \{\underline{0}_n\}$

- if $\underline{u}, \underline{v} \in H$, $\underline{u} + \underline{v}$ is in H
- if $\underline{u} \in H$, $c \in \mathbb{R}$, then $c\underline{u} \in H$.

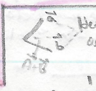
↳ A basis of a subspace is a linearly independent set that spans H. $\beta = [\underline{b}_1 \cdots \underline{b}_n]$ if β basis for H, then $x \in H \Rightarrow x = c_1\underline{b}_1 + \cdots + c_n\underline{b}_n$ $\Updownarrow$

↳ Colspace A = $\text{Span}\{a_1, \ldots a_n\}$ $a_i \in \mathbb{R}^m$, $\text{Col}A$ is subspace of $\mathbb{R}^m$. $\to$ the pivot columns of a matrix form a basis for ColA. ↳ not echelon form. $\quad [\underline{X}]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ coordinates of vector $\underline{X}$ relative to β (basis)

↳ $\text{Nul}A = \{x \in \mathbb{R}^n \mid A\underline{x} = 0\}$ NulA is subspace of $\mathbb{R}^n$.

↳ Dimension — # of vectors in a basis. $\dim H = n$, $\dim\{\underline{0}\} = 0$, $\dim\mathbb{R}^n = n$

↳ Rank = $\dim(\text{col}A)$, $\text{Rank}A + \dim(\text{null}A) = n$ ↳ using basis for column space.



$|\det[\vec{a}\,\vec{b}]|$ = area of parallelogram

usually $A(c\underline{x} + d\underline{y}) \neq (c+d)\underline{b}$

Sol'n's to inhomogeneous eqn's tend not to be subspaces

$A(c\underline{x} + d\underline{y}) = cA\underline{x} + dA\underline{y} = (c+d)\underline{b}$

The basis theorem — H be p dimensional subspace of $\mathbb{R}^n$. Any linearly indp. set of p elements in H is a basis for H. Any set of p elements of H that spans H is a basis for H.

$\to$ a basis for $\mathbb{R}^n$ must contain n vectors

Subspace — set of sol'ns of m homogeneous linear equations w/ n unknowns is subspace of $\mathbb{R}^n$

A_{ij} = ith row, jth column entry.

Similarity — A similar to B iff $- A = PBP^{-1}$

↳ then $\det A = \det B$

↳ characteristic polynomials are $P_A(t) = P_B(t)$

↳ same eigenvalues.

$\Rightarrow$ row equivalence

Cofactor matrix = $\begin{bmatrix} c_{11} & \cdots & c_{n1} \\ \vdots & & \\ c_{1n} & \cdots & c_{nn} \end{bmatrix}$, adjugate matrix = transpose of cofactor matrix, $A^{-1} = \dfrac{1}{\det A}[\text{adj}A]$

Eigenvalues

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ or $A^{-1} = [A \mid I_n] \sim [I_n \mid A^{-1}]$

$\to \lambda$ is an eigenvalue of A iff $(A - \lambda I)\underline{x} = 0$ has eigen vector $\underline{x} \neq \underline{0}_n$

iff $\det(A - \lambda I)\underline{x} = 0 \Rightarrow P_A(t) = \det(A - tI) \to$ characteristic polynomial in degree n of t.

$\to E_\lambda = \{\underline{v} \in \mathbb{R}^n \mid A\underline{v} = \lambda\underline{v}\}$, eigenspace associated to λ, includes eigenvectors and $\underline{0}_n$.

↳ is a subspace of $\mathbb{R}^n$. $\to$ eigenvectors that correspond to distinct eigenvalues are linearly independent.

$\to$ To calculate — if triangular, eigenvalues are its diagonal entries.

- is $\underline{x}$ eigenvector? $\to A\underline{x} = [A][\underline{x}] = \lambda[\underline{x}]$ — is λ an eigenvalue? $\to \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = \underline{0}$ — have nontrivial sol'ns? the sol'ns are the basis of the eigenspace for the eigenvalue in $\mathbb{R}^n$.
- powers of diagonal matrix = $\begin{bmatrix} d^k & 0 \\ 0 & d^k \end{bmatrix} = D^k$ $k \geq 1$. $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$

$\to$ to diagonalize
- 1) Find λ of A, $0 = \det(A - \lambda I) = (\lambda_1 - \lambda)^{m_{\lambda_1}} \cdots (\lambda_n - \lambda)^{m_{\lambda_n}}$
- 2) Find n linearly independent eigenvectors of A. (Find basis for eigenspace of each λ_i ($\underline{V}_{\lambda_i}$)
- 3) Construct P from these eigenvectors, and D from their corresponding eigenvalues, such that $P = [v_{\lambda_1} \cdots v_{\lambda_n}]$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Diagonal matrix — A matrix is diagonalizable if A is similar to a diagonal matrix

↳ $A^k = P^{-1}D^kP$ ↳ $AP = PD$ ↳ a diagonal matrix similar to A.

$\Leftrightarrow$ iff A has n linearly independent eigenvectors.

↳ if A has n distinct eigenvalues, automatically diagonalizable.

↳ 0 matrix is diagonalizable, only has eigenvalue of 0.

$\Leftrightarrow$ A = matrix, D = diagonal matrix w/ λ of A, $P = [v_1 \cdots v_n] \to$ eigenvectors associated w/ each λ.

$P_A(t) = (t - \lambda_1)^{m_{\lambda_1}} \cdots$, $\dim E_{\lambda_i} \leq m_{\lambda_i}$, $\sum m_{\lambda_i} = n$

$\Leftrightarrow$ iff $P_A(t)$ factors into linear terms ie $= (\lambda_1 - t)^{m_{\lambda_1}}(\lambda_2 - t)^{m_{\lambda_2}} \cdots$ ~~ex:~~ $= t(t^2+1)$ and if $\dim E_{\lambda_i} = m_{\lambda_i}$

*can have free variables and be inconsistent.

$[m \times n] \cdot [n \times p] = [m \times p]$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$; $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj+bm+cp & \cdots \\ dj+em+fp & \cdots \\ gj+hm+ip & \cdots \end{bmatrix}$ $\quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}^T$

Elementary Matrix — an invertible, $n \times n$ matrix resulting from performing one elementary row operation on the identity matrix.

↳ row switching — switch row i with row j in I_n ($[1] \leftrightarrow [2] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$) — E^{-1} same as E

↳ row addition — C times row j added to row i, add m to I_n in the ij position ($c[2] + [3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$, E^{-1} is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}$

↳ Scalar multiplication — multiply row i of I_n by scalar c ($c[3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$) ↳ E^{-1} is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

↳ $[A] \underset{E_1}{\sim} \cdots \underset{E_n}{\sim} [I_n] \Rightarrow E_n \cdots E_1A = I_n \Rightarrow A = E_1^{-1} \cdots E_n^{-1}$

$\underset{2\times2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\underset{2\times1}{\begin{bmatrix} e \\ f \end{bmatrix}} = \underset{2\times1}{\begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}}$; $\underset{1\times2}{\begin{bmatrix} a & b \end{bmatrix}}\underset{2\times2}{\begin{bmatrix} c & d \\ e & f \end{bmatrix}} = \underset{1\times2}{\begin{bmatrix} ac+be & ad+bf \end{bmatrix}}$

$\underset{1\times2}{\begin{bmatrix} a & b \end{bmatrix}}\underset{2\times1}{\begin{bmatrix} c \\ d \end{bmatrix}} = \underset{1\times1}{[ac+bd]}$ $\quad A^{-1} = E_n \cdots E_1$

Ax is linear combination of columns of A using corresponding entries in x as weights $A\underline{x} = [\underline{a}_1 \cdots \underline{a}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\underline{a}_1 + \cdots + x_n\underline{a}_n$ $\quad$ matrix eqn $= Ax = b$, vector eqn $= [A \mid b]$ ↓ system of linear equations

$C[1] + [2] = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C[1] + [3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}$

$C[2] + [1] = \begin{bmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$c[3] + [1] = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$c[3] + [2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

area of parallelepiped formed by vectors v_1, v_2, v_3 and the origin $= \det[v_1 v_2 v_3]$; if $\underline{v}$ transformed by T w/ standard matrix A; new area is $\det[v_1 v_2 v_3] \cdot \det A$. ↳ columns of S. $T(S)$

$\hat{\underline{y}} = \text{proj}_L\underline{y} = \dfrac{\underline{y} \cdot \underline{u}}{\underline{u} \cdot \underline{u}}\underline{u}$, $L = \text{span}\{\underline{u}\}$

Gram Schmidt Process — let $\{\underline{x}_1 \cdots \underline{x}_p\}$ basis of subspace $W \neq 0$ in $\mathbb{R}^n$, let $\underline{v}_1 = \underline{x}_1$, $\underline{v}_2 = \underline{x}_2 - \dfrac{(\underline{x}_2 \cdot \underline{v}_1)}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1$, $\underline{v}_3 = \underline{x}_3 - \dfrac{\underline{x}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2}\underline{v}_2 - \dfrac{\underline{x}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1$

$\{\underline{v}_1\,\underline{v}_2,\underline{v}_3\}$ is an orthogonal set *

dot product $\underline{u} \cdot \underline{v} = \underline{u}^T\underline{v} = \sum_{i=1}^{n} u_iv_i$

Norm $= \|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$, $\|c\underline{v}\| = |c|\|\underline{v}\|$

$\text{dist}(\underline{u}, \underline{v}) = \|\underline{u} - \underline{v}\|$

Least squares sol'n — If A $m \times n$ matrix, $b \in \mathbb{R}^m$, a least squares sol'n to $A\underline{x} = \underline{b}$ is a $\hat{\underline{x}} \in \mathbb{R}^n$ such that $\|b - A\hat{\underline{x}}\| \leq \|b - A\underline{x}\|$ $\forall x \in \mathbb{R}^n$. $\hat{b} = \text{proj}_{\text{Col}A}b \in \text{Col}A$, so $A\underline{x} = \hat{\underline{b}}$ consistent, where $\hat{\underline{x}}$ is sol'n, $\underline{z} = b - \hat{b}$ orthogonal to $\text{Col}A$, $\underline{a}_i \cdot \underline{z} = 0 \Rightarrow \underline{a}_i \cdot (\underline{b} - \hat{\underline{b}}) = 0 \Rightarrow \underline{a}_i \cdot (\underline{b} - A\hat{\underline{x}}) = 0 \Rightarrow A^T(\underline{b} - A\hat{\underline{x}}) = 0$

$\Rightarrow A^T\underline{b} - A^TA\hat{\underline{x}} = 0 \Rightarrow A^T\underline{b} = A^TA\hat{\underline{x}}$. Let A be $m \times n$, (A^T $n \times m$): $A\underline{x} = \underline{b}$ has unique least squares sol'n $\forall b \in \mathbb{R}^m \Leftrightarrow$ A linearly indp $\Leftrightarrow A^TA$ invertible, so $\hat{\underline{x}} = (A^TA)^{-1}A^T\underline{b}$ — least squares sol'n has minimum $\hat{b} = b - A\hat{x}$

$\text{proj}_{\text{Col}A}b = \hat{b} = \dfrac{b \cdot a_1}{a_1 \cdot a_1}a_1 + \dfrac{b \cdot a_2}{a_2 \cdot a_2}a_2 + \cdots$ for $[a_1 \cdots a_n]$ orthogonal matrix

$\hat{x} = \langle x_1, x_2, \ldots \rangle$

$(\text{Col}A)^\perp = \text{Nul}(A^T)$

$(\text{Row}A)^\perp = \text{Nul}(A)$

invertible, not diagonalizable $= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

diagonalizable, not invertible $= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\underline{x} = \underline{p} + x_i\underline{v}$ — eqn of line through $\underline{p}$ parallel to $\underline{v}$

If $A = QR$, Q invertible, A similar to $B = RQ$ $\quad A = QR \Rightarrow AQ = QRQ \Rightarrow AQ = QB \Rightarrow A = QBQ^{-1}$

If $\exists A^{-1}$, $A = PBP^{-1}$ show B is invertible. $\quad A = PBP^{-1} \Rightarrow \det A = \det P \det P^{-1} \det B \Rightarrow \det A = \det B \neq 0$

prove $\text{Nul}T = \{0\}$ iff T = one to one, let $T: \mathbb{R}^n \to \mathbb{R}^m$, so $\exists\, u \neq 0$, $T(u) = 0$, but $T(0) = 0 \to$ so contradiction.

If A diagonalizable + invertible, prove A^{-1} is: $A = PBP^{-1} \Rightarrow A^{-1} = (PBP^{-1})^{-1} \Rightarrow A^{-1} = P^{-1}B^{-1}P$ (we know B invertible since A is so has no $\lambda = 0$.

If A similar to B, B similar to C, show A similar to C; $A = PBP^{-1}$, $B = QCQ^{-1} \Rightarrow A = PQCQ^{-1}P^{-1} \Rightarrow A = (PQ)C(PQ)^{-1}$

Echelon form — all nonzero rows above any rows of all 0's. - each leading entry of a row is in a column to the right of the leading entry of the row above it. AKA if a row contains a leading 1, each row above it contains a leading 1 further to the left. - all entries in a column below a leading entry are 0's. $\to$ pivot position — a pivot column is a column that contains a pivot position

reduced echelon form — leading entry in each nonzero row = 1, each leading 1 only nonzero entry in its column. pivot if $\neq 0$ in echelon form

- Each matrix row equivalent to one and only one reduced echelon matrix

Consistent $\Leftrightarrow$ rightmost column of augmented matrix is not a pivot column. $\leftarrow$ Existence

↳ uniqueness — only basic variables = unique sol'n, free variables = many solutions — can find parametric description of sol'n set.

* Similarity implies row equivalence $\to$ can use row operations to make one matrix into another.

$T: \mathbb{R}^3 \to \mathbb{R}^2$

$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x+y \\ y+z \end{bmatrix}$

$A = [T(e_1)\,T(e_2)\,T(e_3)]$

$A = \left[T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$

$A = \left[\begin{bmatrix} 2+0 \\ 0+0 \end{bmatrix}\begin{bmatrix} 0+1 \\ 1+0 \end{bmatrix}\begin{bmatrix} 0+0 \\ 0+1 \end{bmatrix}\right]$

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^3$

$T\begin{bmatrix} x \\ y \end{bmatrix} =$

$T\begin{bmatrix} 3x+y \\ 5x+7y \\ x+3y \end{bmatrix}$

$= \left[T\begin{bmatrix} 1 \\ 0 \end{bmatrix} T\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$

$= \left[\begin{bmatrix} 3+0 \\ 5+0 \\ 1+0 \end{bmatrix}\begin{bmatrix} 0+1 \\ 0+7 \\ 0+3 \end{bmatrix}\right]$

$= \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} = A$

Sol'n to $A\underline{x} = 0$

Sol'n set to $A\underline{x} = \underline{b}$

1 sol'n to $A\underline{x} = \underline{b}$, must assume $A\underline{x} = \underline{b}$ consistent.

- linear transformations
- matrix algebra
- determinants
- Eigenvalues
- Subspaces, basis
- Dot product / Orthogonality
- least squares / projections / gram schmidt

A $n\times n$; $A^2=0$ $\Rightarrow \mathrm{Col}A \subseteq \mathrm{Nul}A$, rank $A \le \frac{n}{2}$
A in null space of A
$n\ge 2$ rank $A \Rightarrow$ rank $A \le \frac{n}{2}$

$A=-A$
n even $\Rightarrow$ $\det A = -\det A$
$\det A = (-1)^n \det A$ $n=\#$ of rows of A, since -1 multiplied to each row.

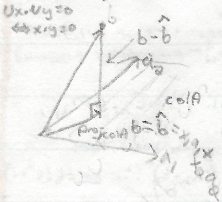
$x^T(A^TAx) = x^T0 = x\cdot 0 \Rightarrow x^TA^TAx = (Ax)^T(Ax) = \|Ax\|=0$ so $Ax=0$
$Ax=0 \Rightarrow A^T(Ax)=A^T(0)=0$ only happens when $x=0$
if A lin indp $\Rightarrow A^TA$ invertible.

Useful Proofs

- is H a subspace? let $u,v\in H=\{v_1,v_2\}$ $u=c_1v_1+c_2v_2$, $v=c_3v_1+c_4v_2 \to u+v=(c_1+c_3)v_1+(c_2+c_4)v_2$
  $\to ku = kc_1v_1+kc_2v_2$
- is b in colspace of A? $Ax=b$ $[A|b]$, solve for x.
- is v in nulA? $Av=0$ has soln?
- $\{v_1\cdots v_n\}$ lin dependent, so let v_p be 1st vector that's linear combination of previous vectors and $v_p=c_1v_1+\cdots+c_{p-1}v_{p-1}$
- if $v_1\cdots v_r$ are eigenvectors to $\lambda_1\cdots\lambda_r$ distinct, then $Av_p=A(c_1v_1)+\cdots+A(c_{p-1}v_{p-1})=c_1\lambda_1v_1+\cdots+c_{p-1}\lambda_{p-1}v_{p-1}$
  and $\lambda_pv_p=\lambda_p(c_1v_1+\cdots+c_{p-1}v_{p-1})=\lambda_pc_1v_1+\cdots+\lambda_pc_{p-1}v_{p-1}=c_1\lambda_1v_1+\cdots+c_{p-1}\lambda_{p-1}v_{p-1} \Rightarrow c_1(\lambda_p-\lambda_1)v_1+\cdots+c_{p-1}(\lambda_p-\lambda_{p-1})v_{p-1} = 0$
  ↳ the weights $c_1\cdots c_{p-1}$ must all be 0, since $v_i\neq 0$ since it's an eigenvector.
- $B=P^{-1}AP \Rightarrow B-\lambda I = P^{-1}AP-\lambda I = P^{-1}AP-\lambda P^{-1}P = P^{-1}(AP-\lambda P)=P^{-1}(A-\lambda I)P \Rightarrow \det(B-\lambda I)=\det P^{-1}\det(A-\lambda I)\det P = \det(A-\lambda I) = \det(B-\lambda I)$
- $AP=PD \Rightarrow AP = Av_1\cdots Av_n = v_1D_{11}\cdots v_nD_{nn} \Rightarrow Av_1=v_1D_{11}\cdots \Rightarrow Av_1=\lambda_1v_1$
- vector subspace $H=\begin{bmatrix}x+y\\x-y\end{bmatrix}$, let $u=\begin{bmatrix}a+b\\a-b\end{bmatrix}$ $v=\begin{bmatrix}c+d\\c-d\end{bmatrix}$ $u+v=\begin{bmatrix}a+b+c+d\\a-b+c-d\end{bmatrix}=\begin{bmatrix}(a+c)+(b+d)\\(a+c)-(b+d)\end{bmatrix}$, $ku=\begin{bmatrix}ka+kb\\ka-kb\end{bmatrix}$ $0=\begin{bmatrix}0+0\\0-0\end{bmatrix}$
- Tools – $A+0$ matrix $=A$; $A(BC)=(AB)C$; $A(B+C)=AB+AC$; $(B+C)A=BA+CA$; $r(AB)=A(rB)=(rA)B$; $I_mA=AI_n$
  $(A^T)^T=A$; $(A+B)^T=A^T+B^T$; $(rA)^T=rA^T$; $(AB)^T=B^TA^T$; $(A^{-1})^{-1}=A$; $(AB)^{-1}=B^{-1}A^{-1}$; $(A^T)^{-1}=(A^{-1})^T$
- linear transformation – Show $T(0_n)=0_m$, $T(cu+dv)=cT(u)+dT(v)$; use $T(x)=Ax$ for $A=[T(e_1)\,T(e_2)\cdots T(e_n)]$ $T:\mathbb{R}^n\to\mathbb{R}^m$ $x\in\mathbb{R}^n$, $b\in\mathbb{R}^m$; 2nd column of I_n
  ↳ one to one $=A$ has pivot in every column(n) $T(x)=b$ has at most one soln for each b. ↳ standard matrix of T. $A=m\times n$ always. $=T(u)=T(v) \Rightarrow u=v$. = columns of A lin. indp.
  ↳ onto $=T(x)=b$ has at least one soln for each b = columns of A span $\mathbb{R}^m$ $=A$ has a pivot in every row (m)
  $\exists A^{-1} \Rightarrow Ax=b \Rightarrow x=A^{-1}b$, if one-to-one, $Ax=0$ only trivial soln.
  $x=I_nx \Rightarrow x=e_1x_1+\cdots+e_nx_n \Rightarrow T(x)=T(e_1x_1+e_nx_n) \Rightarrow x_1T(e_1)+\cdots+x_nT(e_n) \Rightarrow [T(e_1)\cdots T(e_n)]\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} \Rightarrow Ax$
  IMT $\to A^{-1}$ exists $\Leftrightarrow A\sim I_n \Leftrightarrow n$ pivot positions $\Leftrightarrow Ax=0$ only trivial soln $\Leftrightarrow$ columns A lin ind $\Leftrightarrow x\to Ax$ one to one $\Leftrightarrow Ax=b$ one soln for each b
  $\Leftrightarrow$ columns A span $\mathbb{R}^n \Leftrightarrow x\to Ax$ onto $\Leftrightarrow AC=I_n \Leftrightarrow DA=I_n \Leftrightarrow A^T$ invertible $\Leftrightarrow \det A\neq 0 \Leftrightarrow 0$ is not an eigenvalue of A.
  $A(v+u)=Au+Av$, $A(cu)=c(Au)$; $w=p+v_h$, if lin indp... $A(w)=A(p+v_h)=Ap+A(v_h)=A(p)+0=A(p)=b$ ↳ soln to $Ax=0$ so $Av_h=0$ ↳ unique soln since lin indp...
- to prove a basis $\Rightarrow$ a linearly independent set of n vectors is a basis for subspace of $\mathrm{Dim}=n$ ↳ show its linearly indp, and that there is n of them.

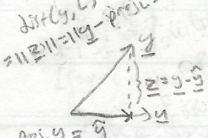
Theorem – For each $b\in\mathbb{R}^m$, $Ax=b$ has a soln $\Leftrightarrow$ Each $b\in\mathbb{R}^m$ is a linear combination of the columns of A. $\Leftrightarrow$ columns of A span $\mathbb{R}^m$ $\Leftrightarrow$ A has a pivot position in each row.

linear dependent – if $\{v_1\cdots v_p\}$ $\exists c_1,\ldots,c_p \sim \forall c_i=0$ $c_1v_1+\cdots+c_pv_p=0 \to$ at least one vector linear combination of the others, $\{v_1\cdots v_p\}\in\mathbb{R}^n$, $p>n$
- if $0\in\{v\cdots v_p\}$
- product of n invertible matrices is invertible.
- For any $u,v\in\mathbb{R}^n$: $u\cdot v=v\cdot u$; $(u+v)\cdot w=u\cdot w+v\cdot w$; $(cu)\cdot v=c(u\cdot v)$; $u\cdot u\ge 0$, $u\cdot u=0 \Leftrightarrow u=0$. $v\cdot v=\|v\|^2$
- orthogonal $\Leftrightarrow \mathrm{dist}(u,v)=\mathrm{dist}(u,-v)$, $u\cdot v=0$, $\|u+v\|^2=\|u\|^2+\|v\|^2$; $\|u+v\|^2=(u+v)\cdot(u+v)=u\cdot u+v\cdot v+2u\cdot v = \|u\|^2+\|v\|^2+0$
  ↳ let w be a subspace of $\mathbb{R}^n$, $w=\mathrm{Span}\{w_1\cdots w_n\}$. Then $w^\perp=\{x\in\mathbb{R}^n \mid x\cdot v=0\ \forall v\in w\}$
  ↳ $w^\perp$ is a subspace of $\mathbb{R}^n$, only common point between w and $w^\perp$ is the origin (0).
  ↳ orthogonal set – set of vectors $\{v_1\cdots v_m\}$ if $v_i\cdot v_j=0$ whenever $i\neq j$. – Orthonormal if orthogonal + $\|v_i\|=1\ \forall v_i$
  ⇒ linearly independent. → proof: Suppose $0=c_1v_1+\cdots+c_mv_m$, consider $i=1$, $0\cdot v_1=(c_1v_1+\cdots+c_mv_m)\cdot v_1=c_1v_1v_1+\cdots+c_mv_mv_1=c_1(v_1\cdot v_1)=0$ so $c_i=0$ so linearly indp, $=0$ since orthogonal set, $c_1=0$ since $c_1\neq 0$ so $c_1=0$
  ↳ orthogonal basis is a basis for a subspace that is also an orthogonal set. If $y\in w$, for $w=$ an orthogonal basis $=\mathrm{span}\{v_1\cdots v_p\}$,
  $y=c_1v_1+\cdots+c_pv_p \Rightarrow y\cdot v_1=(c_1v_1+\cdots+c_pv_p)\cdot v_1=c_1(v_1\cdot v_1)+\cdots+\underbrace{c_p(v_p\cdot v_1)}_{=0}=c_1(v_1\cdot v_1)=y\cdot v_1 \Rightarrow c_1=y\cdot v_1/v_1\cdot v_1 \Rightarrow c_1=y\cdot v_1/\|v_1\|^2$



m < n cannot be one-to-one.
m > n cannot be onto.
m×n lin ind ⇔ n pivot columns.

Orthogonal Matrices
↳ square invertible matrix
↳ $U^{-1}=U^T$
↳ orthonormal columns
↳ Also has orthonormal rows too.
↓ *
A matrix has orthonormal columns $\Leftrightarrow U^TU=I \to$ then $\|Ux\|=\|x\|$, $Ux\cdot Uy=x\cdot y$, $Ux\cdot Uy=0 \Leftrightarrow x\cdot y=0$



$\mathrm{proj}_Ly=\hat{y}$ ↳ $L=\mathrm{span}\{u\}$
projection same regardless of length of vector it's being projected onto.

Each $y\in\mathbb{R}^n$ can be uniquely written as $y=\hat{y}+z$, where $\hat{y}=cu$, for $c\in\mathbb{R}$ and $w\cdot u=0$ ↳ $z=y-\hat{y}$. $\hat{y}\in w$, $z\in w^\perp$. If w is an orthogonal basis, $\hat{y}=\sum_{i=1}^{p}\frac{y\cdot u_i}{\|u_i\|^2}$
$0=w\cdot u \Rightarrow (y-\hat{y})\cdot u=(y-cu)\cdot u=y\cdot u-c(u\cdot u)=0$
$y\cdot u=(u\cdot u)c \Rightarrow c=y\cdot u/\|u\|^2$
So $\hat{y}=c\cdot u=\frac{y\cdot u}{\|u\|^2}u$

- $\frac{y\cdot(ku)}{\|ku\|^2}(ku)=\frac{k^2(y\cdot u)}{k^2\|u\|^2}u=\hat{y}$
- let A be $m\times n$ matrix and $\{e_1\cdots e_n\}$ orthonormal set $\Leftrightarrow A^TA=I \Rightarrow \|Ax\|=\|x\| \Rightarrow \|Ax\|=\sqrt{(Ax)\cdot(Ax)}=\sqrt{(Ax)^T(Ax)}=\sqrt{x^TA^TAx}=\sqrt{x^Tx}=\sqrt{x\cdot x}=\|x\|$
  $Ax\cdot Ay=x\cdot y \Rightarrow Ax\cdot Ay=(Ax)^T(Ay)=x^TA^TAy=x^Ty=x\cdot y$.
  $Ax\cdot Ay=0 \Leftrightarrow x\cdot y=0$
- $z\cdot u=(y-\hat{y})u_i=y\cdot u_i-\hat{y}\cdot u_i=y u_i-\left(\sum_{j=1}^{p}\frac{y\cdot u_j}{u_j\cdot u_j}u_j\right)u_i$
  $=\frac{y\cdot u_i}{}-\frac{y\cdot u_i}{u_i\cdot u_i}u_i\cdot u_i=yu_i-y\cdot u_i=0$ and $y=\hat{y}+(y-\hat{y})=\hat{y}+z$

If $y\in\mathrm{span}\{u_1\cdots u_p\}$, proj_wy is best approximation of y by elements in $w \Rightarrow \|y-\hat{y}\|<\|y-v\|\ \forall v\in w$. If $U=[u_1\cdots u_p]$, if orthonormal basis for w subspace, then $\mathrm{proj}_wy=UU^Ty\ \forall y\in\mathbb{R}^n \to$ orthogonal projection of y onto column space of U.

$\hat{x}$ is a least squares soln if $\hat{x}$ continues...
$\Rightarrow A^TAx=A^Tb \Rightarrow 0=A^Tb-A^TAx \Rightarrow 0=A^Tb-A^TA\hat{x} \Rightarrow 0=A^T(b-A\hat{x}) \Rightarrow 0=A\cdot(b-A\hat{x}) \to A\perp(b-A\hat{x})$, since $y=\hat{y}+z$, $\hat{y}\perp z$ is unique decomposition, we know $Ab=A\hat{x}+(b-A\hat{x})$ is unique, and $Ax=\mathrm{proj}_{\mathrm{Col}A}b\in\mathrm{col}A$ and $b-A\hat{x}=\mathrm{proj}_{(\mathrm{Col}A)^\perp}b\in(\mathrm{Col}A)^\perp$ so $\hat{x}$ is the least squares soln.
least squares solution = find an x that makes Ax as close as possible to b, $b\notin\mathrm{col}A$, an $\hat{x}$ that satisfies $A\hat{x}=\hat{b}$ where $\hat{b}$ is orthogonal projection of b onto $\mathrm{Col}A$.

- $\mathrm{Nul}A=\mathrm{Nul}A^TA \Rightarrow$ if A lin, ind columns, A^TA invertible.

orthogonal decomposition theorem – let w subspace of $\mathbb{R}^n$, then each $y\in\mathbb{R}^n$ can be written uniquely as $y=\hat{y}+z$ for $\hat{y}\in w$, $z\in w^\perp$. If $\{u_1\cdots u_n\}$ orthogonal basis of w, then $\hat{y}=\frac{y\cdot u_1}{u_1\cdot u_1}u_1+\cdots+\frac{y\cdot u_p}{u_p\cdot u_p}u_p$ and $z=y-\hat{y}$

Suppose 2 soln's to $Ap=b$, $p_1,p_2\in\mathrm{row}A$, so $A(p_1-p_2)=Ap_1-Ap_2=b-b=0 \Rightarrow p_1-p_2\in\mathrm{Nul}A$ so $p_1=\underbrace{p_2}_{\in\mathrm{row}A}+\underbrace{(p_1-p_2)}_{\to\mathrm{Nul}A}$
$p_2=\underbrace{p_1}_{\mathrm{row}A}+\underbrace{(p_2-p_1)}_{\to\mathrm{Nul}A}$, $p_1=\underbrace{p_1}_{\mathrm{row}A}+\underbrace{0}_{\to\mathrm{Nul}A}$ So by ODT $p_1=p_2$ since decomposition must be unique ■